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SOME GENERALIZATIONS OF GEODESICS*

BY

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1. Introduction

Let $y^{(k)}$ $(k = 1, 2, 3, 4)$ be four linearly independent analytic functions of two independent variables, u and v, and interpret $y^{(1)}, \ldots, y^{(4)}$ as the homogeneous coördinates of a point P_y in ordinary space. If the locus of P_y is a proper surface S, and if the curves u = const. and v = const. are the asymptotic lines of S, assumed to be distinct, $y^{(1)}, \ldots, y^{(4)}$ will form a fundamental system of solutions of a system of linear homogeneous differential equations of the form

$$(1) \qquad \begin{aligned} y_{uu} + 2ay_u + 2by_v + cy = 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y = 0, \end{aligned}$$

where $a, b, \ldots, c'$ are analytic functions of u and v, which must satisfy certain integrability conditions, one of which is

$$(1a) \qquad a_v = b'_u.$$

If the coördinates $y^{(1)}, \ldots, y^{(4)}$ are homogeneous cartesian coördinates:

$$(2) \qquad y^{(1)} = x, \; y^{(2)} = y, \; y^{(3)} = z, \; y^{(4)} = 1,$$

we shall have

$$(2a) \qquad \begin{aligned} 2a &= -\left\{ {11 \atop 1} \right\}, \quad 2b = -\left\{ {11 \atop 2} \right\}, \quad c = 0, \\ 2a' &= -\left\{ {22 \atop 1} \right\}, \quad 2b' = -\left\{ {22 \atop 2} \right\}, \quad c' = 0, \end{aligned}$$

where the Christoffel symbols are formed from the quadratic form

$$(3) \qquad ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

for the square of the arc element of S.

We propose to study the properties of certain two-parameter families of curves on S, namely those which are defined by a differential equation of the second order of the form

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$$(4) \qquad v'' = A(v')^3 + 3B(v')^2 + 3Cv' + D, \quad v' = \frac{dv}{du}, \quad v'' = \frac{d^2v}{du^2},$$

where A, B, C, D are analytic functions of u and v. In this paper, all such curves shall be called *hypergeodesics*. Moreover, all hypergeodesics which satisfy the same equation of form (4) shall be said to belong to the same *family*.

If we are using cartesian coördinates, these curves will reduce to *geodesics*, i. e., to extremal curves of the integral

$$s = \int \sqrt{Edu^2 + 2Fdudv + Gdv^2},$$

if the coefficients $A, \ldots, D$, are expressible in the form

$$(5) \qquad A = \left\{ {22 \atop 1} \right\}, 3B = 2\left\{ {12 \atop 1} \right\} - \left\{ {22 \atop 2} \right\}, 3C = -2\left\{ {12 \atop 2} \right\} + \left\{ {11 \atop 1} \right\}, D = -\left\{ {11 \atop 2} \right\}.$$

If we consider any other integral of the form

$$(6) \qquad s' = \int \sqrt{E'du^2 + 2F'dudv + G'dv^2},$$

its extremal curves will also be hypergeodesics, satisfying an equation of form (4) with the coefficients

$$(7) \quad A' = \left\{ {22 \atop 1} \right\}', 3B' = 2\left\{ {12 \atop 1} \right\}' - \left\{ {22 \atop 2} \right\}', 3C' = -2\left\{ {12 \atop 2} \right\}' + \left\{ {11 \atop 1} \right\}', D' = -\left\{ {11 \atop 2} \right\}',$$

the new Christoffel symbols $\left\{ {ij \atop k} \right\}'$ being formed with respect to the quadratic form

$$(7a) \qquad ds'^2 = E'du^2 + 2F'dudv + G'dv^2.$$

Since (4) contains four arbitrary functions of u and v as coefficients, while the coefficients (7) depend on only three arbitrary functions, E', F', G', those equations of form (4), whose coefficients are expressible in the form (7), define only a special class of hypergeodesics, which we shall call *quadratic extremal curves*. If we change from the surface S to a new surface S' whose squared arc element is given by (7*a*), these curves will become geodesics on S'.

Through a given point P of a surface pass ∞^1 geodesics, one for every tangent of P. Their osculating planes form a pencil with the surface normal of P as axis. For this reason we shall say that the congruence of normals is axially related to the surface. Let us consider *any congruence axially related to the surface*, so that one line l' of the congruence will pass through every point P of the surface without, however, being tangent to the surface at P. Such a congruence will determine a two-parameter family of curves on S, such that the osculating planes of the ∞^1 curves of the family which pass through P will form a pencil

with l' as axis. Such curves shall henceforth be called *axial union curves* of the surface with respect to the congruence Γ' formed by the lines l'.* Their differential equation is also of form (4), the coefficients being subject to the conditions

$$A = -2a' = \left\{ {22 \atop 1} \right\}, \quad D = 2b = -\left\{ {11 \atop 2} \right\}. \tag{8}$$

Of course, a geodesic, being a quadratic extremal curve, as well as an axial union curve, satisfies both conditions (7) and (8).

Dualistic considerations lead to the notion of congruences *radially* related to a surface and *radial* union curves. *A congruence* Γ *is radially related to a surface* if one of its lines l is assigned to every tangent plane π of the surface without however being tangent to the surface at P. Instead of thinking of a curve C on S as a point locus, let us think of the one-parameter family of planes tangent to S along C, or their developable D, whose generators will be the tangents t' of S conjugate to the tangents t of C.

To the osculating plane of a point P of C will correspond, in this dualistic view, the point in which the corresponding conjugate tangent t' of P meets the cuspidal edge of D. This point has been used extensively so far only in the theory of conjugate nets and is there called a Laplace transform of P. In that theory, however, there arises a second point of this kind which is on t, and the line joining these two points is called the *ray of* P. For this reason we shall henceforth call the point in which the tangent t', conjugate to the tangent t of C at P, meets the cuspidal edge of D, the *ray point of* P with respect to the curve C. By analogy the osculating plane of C at P will sometimes be called the *axis plane* of P, for the line of intersection of the osculating planes for the curves C and C' of a conjugate net is called the *axis* of P.

To axial union curves will correspond, in this dualistic view, *radial union curves whose defining property is as follows; the ray point of each of its points lies on the corresponding line l of a radially related congruence.* Equation (4) will define radial union curves on S if and only if

$$A = 2a' = -\left\{ {22 \atop 1} \right\}, \quad D = -2b = \left\{ {11 \atop 2} \right\}\dagger. \tag{9}$$

The curves of a pencil of conjugate nets‡ form a two-parameter family given by

* In Miss Sperry's thesis they are called *union curves*. Pauline Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1918), pp. 213–224. See also G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences.*, these Transactions, vol. 20 (1919) pp. 79–153.

† In Miss Sperry's thesis and in Green's memoir, quoted above, these curves are called *adjoint* or *dual* union curves. To obtain conditions (9) compare (4) and equation (132) of Green's memoir.

‡ E. J. Wilczynski, *Geometrical significance of isothermal conjugacy of a net of curves*, American Journal of Mathematics, vol. 42 (1920), p. 217.

$$\left(\frac{dv}{du}\right)^2 = k\mu,$$

where μ may be any function of u and v, and where k is an arbitrary constant. The second order differential equation of the curves of this family is

$$v'' = \frac{1}{2}\frac{\mu_v}{\mu}v'^2 + \frac{1}{2}\frac{\mu_u}{\mu}v',$$

which is again a special case of (4) characterized by the conditions

$$A = D = 0, \quad B_u = C_v. \tag{10}$$

We have given a number of instances in which differential equations of form (4) arise. They clearly form an important class. This paper is devoted to a discussion of their most striking geometric properties.

2. Properties of the osculating planes

Let C be an integral curve of equation (4) on a surface S defined by (1). The coördinates $y^{(k)}$, of a point P of C, may then be regarded as functions of u, and we find

$$\begin{aligned} \frac{dy}{du} &= y' = y_u + y_v v', \\ \frac{d^2y}{du^2} &= y'' = y_{uu} + 2y_{uv}v' + y_{vv}v'^2 + y_v v'', \end{aligned} \tag{11}$$

or, making use of (1),

$$y'' = -(c + c'v'^2)y - 2(a + a'v'^2)y_u + [-2(b + b'v'^2) + v'']y_v + 2v'y_{uv}. \tag{12}$$

If we substitute into (11) and (12) in succession the four solutions $y^{(k)}$ of (1), we obtain four points y, y', y'' whose plane is the osculating plane of C at P. Let us use the four points defined by y, y_u, y_v, y_{uv} as elements of a local coördinate system in the sense that the local coördinates of a point shall be proportional to x_1, x_2, x_3, x_4 if that point is defined by an expression of the form

$$x_1 y + x_2 y_u + x_3 y_v + x_4 y_{uv}.$$

Then *the equation of the plane which osculates the curve C at P will be*

$$2v'^2x_2 - 2v'x_3 + [v'' + 2(a'v'^3 - b'v'^2 + av' - b)]x_4 = 0, \tag{13}$$

referred to the local tetrahedron of y, y_u, y_v, y_{uv}. So far we have made no use of (4). For an integral curve of (4), the coördinates of the osculating plane are

$$(14)\quad u_1=0,\qquad u_2=2v'^2,\qquad u_3=-2v',$$
$$u_4=(A+2a')v'^3+(3B-2b')v'^2+(3C+2a)v'+D-2b,$$

if the curve passes through P in the direction determined by v'. Since v' is arbitrary we obtain a one-parameter family of such planes, and (14) shows that their envelope is, in general, a cone of class three, called the *axis-plane cone* or *osc-cone* of P. The class reduces to two if either $D-2b$ or $A+2a'$ is equal to zero (but not both). The class reduces to one, so that the cone becomes a straight line, if and only if

$$D-2b=A+2a'=0$$

in accordance with the conditions (8), already noted, for an axial union curve.

If we eliminate v' from (14) we find the equation of the axis-plane cone in plane coördinates, viz.:

$$(15)\quad 2u_2u_3u_4+(A+2a')u_2^3-(3B-2b')u_2^3u_3$$
$$+(3C+2a)u_2u_3^2-(D-2b)u_3^3=0,\qquad u_1=0.$$

The cone has a double tangent plane, namely the plane $u_4=0$, which is tangent to S at P. This plane touches the cone along the elements $u_2=u_4=0$ and $u_3=u_4=0$, which are the asymptotic tangents of S at P. Consequently the cone is a surface of the *fourth order* and must possess three cuspidal tangent planes. The Hessian of (15) is equivalent to

$$(16)\quad 2u_2u_3u_4-3(A+2a')u_2^3-(3B-2b)u_2^2u_3+(3C+2a)u_2u_3^2+3(D-2b)u_3^3=0,$$

so that the coördinates of the three cuspidal tangent planes must satisfy

$$(17)\qquad (A+2a')u_2^3-(D-2b)u_3^3=0$$

as well as (15). But since (15) may be written

$$(18)\quad u_2u_3[2u_4-(3B-2b')u_2+(3C+2a)u_3]+(A+2a')u_2^3-(D-2b)u_3^3=0,$$

the coördinates of the three cuspidal tangent planes are given by

$$(19)\qquad \begin{aligned} &u_1=0,\qquad (A+2a')u_2^3-(D-2b)u_3^3=0,\\ &u_4+(b'-\tfrac{3}{2}B)u_2+(a+\tfrac{3}{2}C)u_3=0.\end{aligned}$$

Consequently these three planes intersect in the line

$$(20)\qquad u_1=0,\ u_4+(b'-\tfrac{3}{2}B)u_2+(a+\tfrac{3}{2}C)u_3=0,$$

which shall be called the *cusp-axis* of P with respect to the given two-parameter family of curves. The cusp-axis may be determined analytically as the line which joins the points y and

$$y_{uv}-(\tfrac{3}{2}B-b')y_u+(\tfrac{3}{2}C+a)y_v. \tag{21}$$

The cusp-axes of all of the points of S form a congruence axially related to S, called the *cusp-axis congruence* of the given two-parameter family of curves. The developables of this congruence correspond to a net of curves on S called the cusp-axis curves, which may be found most conveniently by making use of some general formulas developed by Green.* It should be noted, however, that Green's formulas do not apply directly to a general system of form (1) but only to the canonical form of such a system. If we put

$$y=l(u,v)\bar{y}, \tag{22a}$$

where

$$l_u=-al,\ \ l_v=-b'l, \tag{22b}$$

conditions which are consistent on account of (1a), system (1) is transformed into its canonical form

$$\bar{y}_{uu}+2b\bar{y}_v+f\bar{y}=0,\quad \bar{y}_{vv}+2a'\bar{y}_u+g\bar{y}=0, \tag{22c}$$

where

$$f=c-a_u-a^2-2bb',\quad g=c'-b'_v-b'^2-2aa'. \tag{22d}$$

This transformation may be applied to our equations and produces a result equivalent to putting a and b' equal to zero, and replacing c and c' by f and g, respectively. We find, in this way, *the differential equation of the cusp-axis curves,*

$$\begin{aligned}(f+2b_v-3bB+\tfrac{9}{4}C^2-\tfrac{3}{2}C_u)du^2-\tfrac{3}{2}(B_u+C_v)dudv\\ -(g+2a'_u+3a'C+\tfrac{9}{4}B^2+\tfrac{3}{2}B_v)dv^2=0.\end{aligned} \tag{23}$$

They form a conjugate net, if and only if

$$B_u+C_v=0.$$

The three cuspidal tangent planes of the axis-plane cone intersect the tangent plane π, of P, in the three tangents

$$u_1=0,\ \ (A+2a')u_2^3-(D-2b)u_3^3=0.$$

If $A=D=0$, as is the case if the two-parameter family consists of the curves of a pencil of conjugate nets, these three tangents coincide with the Segre tangents of the point P.† The same thing takes place, more generally, whenever

$$A=\omega a',\quad D=-\omega b, \tag{24}$$

where ω may be an arbitrary function of u and v.

* These Transactions, vol. 20 (1919), p. 90.

† We shall give a new definition for the Segre and Darboux tangents in §4.

Most of these results are due to Fubini.* We sum up the most important ones in the following theorem.

Consider any family of hypergeodesics on a surface S. Those which pass through a given point P, of S, form a one-parameter family. Their osculating planes, at P, envelop a quartic cone of class three, called the axis-plane cone or the osc-cone of P, which touches the tangent plane π along its asymptotic tangents, so that π is a double tangent plane of the cone. The osc-cone possesses three cuspidal tangent-planes which intersect along a line called the cusp-axis of P. Since the four functions, A, B, C, D, of u and v, may be chosen at pleasure, any system of quartic cones, associated with the given surface in this manner, may be regarded as corresponding to some equation of form (4). *Consequently the properties mentioned in this theorem are characteristic of a family of hypergeodesics.*

To every value of v' there corresponds a tangent t of S at P. The tangent t', which corresponds to $-v'$, is conjugate to t. The osculating plane of the corresponding hypergeodesic of the family (4) will be obtained from (14) by changing v' into $-v'$, giving

$$v_1=0, \qquad v_2=2(v')^2, \qquad v_3=2v',$$
$$v_4=-(A+2a')(v')^3+(3B-2b')(v')^2-(3C+2a)v'+D-2b$$

as the coördinates of this plane. If $x_1, \ldots, x_4$ are the coördinates of a point on the intersection of these two planes, we find

$$\rho x_2=-(3B-2b')(v')^2+D-2b,$$
$$\rho x_3=(A+2a')(v')^4+(3C+2a)(v')^2, \qquad \rho x_4=2(v')^2,$$

while x_1 remains arbitrary. Elimination of v' from these equations gives

$$[2x_2+(3B-2b')x_4][2x_3-(3C+2a)x_4]+(A+2a')(D-2b)x_4^2=0, \tag{25}$$

whence follows the following theorem.†

Let the two hypergeodesics of a family, which touch two conjugate tangents of a surface point P, be called conjugate hypergeodesics, and let the line of intersection of their osculating planes be called their axis. Then, the axes of all such pairs of conjugate hypergeodesics, for the same point P, form a quadric cone (25), *called the axis cone, which intersects the tangent plane π of P, in its asymptotic tangents, in such a way that the planes tangent to the cone along the asymptotic tangents intersect in the cusp-axis,* (20), *of P.*

* G. Fubini, *Fondamenti della geometria proiettivo-differenziale di una superficie*, Atti della Reale Accademia delle Scienze di Torino, vol. 53 (1918), p. 1037.

† This is a generalization of a corresponding theorem of Lane's concerned with a pencil of conjugate nets.

This quadric cone bears a very special relation to the surface S at P. The same thing is true of the quadric cone to which the osc-cone at P reduces when $(D-2b)(A+2a')$ is equal to zero. We may, instead, associate, with every point P of S, an arbitrary guadric cone

$$u_1=0, \quad \sum C_{ik}u_iu_k=0 \qquad (i, k=2, 3, 4),$$

having the point P as vertex, thus defining a certain complex whose lines are the elements of these ∞^2 cones. We may then define a two-parameter family of curves on S, *union curves* with respect to this complex, by demanding that the osculating plane ω of any point P of such a curve shall touch the corresponding cone. The differential equation of these curves will be

$$\begin{aligned} &C_{44}[v''+2a(v')^3-2b'(v')^2+2a'v'-2b]^2 \\ &\quad +4v'(C_{42}v'-C_{34})[v''+2a(v')^3-2b(v')^2+2a'v'-2b] \\ &\quad +4C_{22}(v')^4-8C_{23}(v')^3+4C_{33}(v')^2=0. \end{aligned}$$

This equation is *quadratic* in v'', and reduces to the form (4) only if $C_{42}=C_{44}=0$.

3. Application of Segre's correspondence

Toward the end of §1 we discussed very briefly the dualistic correspondence between the osculating plane ω of a curve C at P, and the corresponding ray-point of P. This correspondence was first studied in detail by Segre* and we shall hereafter speak of it as *Segre's correspondence*. To find the equations of this correspondence we observe that the homogeneous coördinates of the ray-point R of P, with respect to a curve C passing through P, are given by†

$$\rho x_1=v''+2(-a'v'^3-b'v'^2+av'+b), \quad \rho x_2=2v', \quad \rho x_3=-2v'^2, \quad \rho x_4=0. \tag{26}$$

On the other hand the coördinates of ω, the osculating plane of C at P, are given by (13). The equations of Segre's correspondence are obtained from (13) and (26) by eliminating v' and v''. They are

$$\rho x_1=u_2u_3u_4+2a'u_2^3+2bu_3^3, \quad \rho x_2=-u_2u_3^2, \quad \rho x_3=-u_2^2u_3, \quad \rho x_4=0, \tag{27}$$

and

$$\sigma u_1=0, \quad \sigma u_2=-x_2x_3^2, \quad \sigma u_3=-x_2^2x_3, \quad \sigma u_4=x_1x_2x_3+2bx_2^3+2a'x_3^3, \tag{28}$$

where ρ and σ are factors of proportionality.

* Segre, *Complementi alla teoria delle tangenti coniugate di una superficie*, Rendiconti della R. Accademia dei Lincei, Series 5, vol. 17 (1908), pp. 405–412.

† Slightly generalized from the formulas given in Green's memoir (these Transactions, vol. 20 (1919), p. 131).

This *cubic birational correspondence* is very convenient for the purpose of developing the formulas which correspond to those of §2 by duality. To the quartic cone of class three corresponds a cubic curve of class four in the tangent plane, namely

$$(29)\qquad \begin{aligned} &x_4=0,\\ &2x_1x_2x_3+(D+2b)x_2^3-(3C+2a)x_2^2x_3+(3B-2b')x_2x_3^2-(A-2a')x_3^3=0, \end{aligned}$$

as the locus of the ray-points of those curves, of the family defined by (4), *which pass through* P. *This cubic curve has three points of inflection which are on the line*

$$(30)\qquad x_4=0,\ 2x_1-(3C+2a)x_2+(3B-2b')x_3=0,$$

which is called the flex-ray of P. Since the expressions

$$(31)\qquad y_u+(\tfrac{3}{2}C+a)y,\qquad y_v-(\tfrac{3}{2}B-b')y$$

represent the points of intersection of the asymptotic tangents of P with this flex-ray, we see by comparing with (21), that *the cusp-axis and the flex-ray of* P *with respect to the two-parameter family of curves defined by* (4) *are Green reciprocals of each other.**

If $D+2b=A-2a'=0$, the cubic curve (29) reduces to three straight lines, the asymptotic tangents of P, and the corresponding flex-ray. This corresponds to the case when equation (4) defines a two-parameter family of radial union curves.

The congruence formed by the flex-rays of all of the points of S is radially related to S. The curves on S which correspond to its developables, *the flex-ray curves of* S, *are determined by the differential equation*

$$(32)\qquad \begin{aligned} &(f-3bB+\tfrac{9}{4}C^2-\tfrac{3}{2}C_u)du^2-\tfrac{3}{2}(B_u+C_v)dudv\\ &\qquad\qquad -(g-3a'C+\tfrac{9}{4}B^2+\tfrac{3}{2}B_v)dv^2=0. \end{aligned}$$

The three points of inflection of the ray-point cubic satisfy equations (30) and

$$(D+2b)x_2^3-(A-2a')x_3^3=0.$$

When the two-parameter family of curves defined by (4) consists of the curves of a pencil of conjugate nets, $A=D=0$, and the above equation shows that the three points of inflection are on the three Darboux tangents of P. The same thing will be true more generally under the conditions (24).

Thus we see that *the cuspidal tangent planes of the axis-plane cone will intersect the tangent plane of* P *in its Segre tangents if the three points of inflection of the ray-point cubic are on the Darboux tangents and conversely*, except in the cases

* This relation will be discussed more in detail in §4.

in which the axis-plane cone or the ray-point cubic reduce to loci of the second or lower degree.

The ray-points of two conjugate tangents determine, as their line of junction, *the ray of the point* P with respect to a given pair of conjugate hypergeodesics. The coördinates of such a ray, corresponding to the pair $(v', -v')$ are

$$u_1 = -2v'^2,\ u_2 = (A-2a')v'^4 + (3C+2a)v'^2,\ u_3 = -(3B-2b')v'^2 - (D+2b),$$

while u_4 remains arbitrary. By eliminating v', we find

$$(33a) \qquad [2u_2 + (3C+2a)u_1]\,[2u_3 - (3B-2b')u_1] + (A-2a)\,(D+2b)u_1^2 = 0,$$

the equation of the *ray conic* in plane coördinates, whence we may deduce

$$(33b) \quad [2x_1 - (3C+2a)x_2 + (3B-2b')x_3]^2 + 4(A-2a')(D+2b)x_2x_3 = 0,\ x_4 = 0,$$

the equations of the ray conic in point coördinates. We have the following theorem.

The ray-points of a pair of conjugate hypergeodesics through a given point P determine the ray of the pair. The envelope of the rays of all such pairs, for a given point P, is a conic called the ray conic of P with respect to the given family of hypergeodesics. This conic is tangent to the asymptotic tangents of P; and the polar of P with respect to the conic coincides with the flex-ray of P The ray conic touches the ray-point cubic in three points which are on the three tangents conjugate respectively to the three tangents which contain the points of inflection of the ray-point cubic.

4. Relation between Segre's and Green's correspondences

Two lines, l and l', are called Green reciprocals of each other with respect to the asymptotic net of a surface S, if l is a line not containing P of the tangent plane π of a point P of S, if l' passes through P but does not lie in its tangent plane, and if l and l' are reciprocal polars of each other with respect to the osculating quadric Q of P. If l is the line of junction of the two points

$$(34a) \qquad y_u - \beta y,\ y_v - \alpha y,$$

l' will be the line which joins y to

$$(34b) \qquad z = y_{uv} - \alpha y_u - \beta y_v.$$

The equations

$$(35) \qquad x_2 + \alpha x_4 = 0,\ \ x_3 + \beta x_4 = 0,$$

represent two planes through l', so that

$$\lambda x_2 + \mu x_3 + (\lambda\alpha + \mu\beta)x_4 = 0$$

will represent any plane through l'. This plane has the coördinates

$$(0, \lambda, \mu, \lambda\alpha + \mu\beta).$$

The points which correspond to the planes of the pencil whose axis is l', in Segre's correspondence, are given by

$$\rho x_1 = \lambda\mu(\alpha\lambda + \beta\mu) + 2a'\lambda^3 + 2b\mu^3, \quad \rho x_2 = -\lambda\mu^2, \quad \rho x_3 = -\lambda^2\mu, \quad \rho x_4 = 0.$$

If we eliminate λ and μ, we find that *the plane cubic curve*

$$(x_1 + \beta x_2 + \alpha x_3)x_2x_3 + 2(bx_2^3 + a'x_3^3) = 0, \quad x_4 = 0 \tag{36}$$

corresponds to the straight line (35) *in Segre's correspondence. It has a double point at P and the asymptotic tangents of P as double point tangents. Its three points of inflection are given by*

$$x_4 = 0, \quad bx_2^3 + a'x_3^3 = 0, \quad x_1 + \beta x_2 + \alpha x_3 = 0. \tag{37}$$

Consequently *these points of inflection are on the Darboux tangents of P.* We may regard this fact as a definition of the Darboux tangents. It is, in fact, simpler than any of the properties of the Darboux tangents which have so far been used for defining them. Since the three points of inflection are also on the line

$$x_1 + \beta x_2 + \alpha x_3 = 0,$$

which is the Green reciprocal of l', we obtain the following additional result.

The Green reciprocal of a line l' through P coincides with the flex-ray of the cubic curve which corresponds to l' in Segre's correspondence.

In similar fashion we find that *in the Segre correspondence, the cone*

$$u_1 = 0, \quad (u_4 - \alpha u_2 - \beta u_3)u_2u_3 + 2(a'u_2^3 + bu_3^3) = 0$$

corresponds to the line l, whose equations are

$$x_1 + \beta x_2 + \alpha x_3 = 0, \quad x_4 = 0.$$

This cone touches the tangent plane of P along the asymptotic tangents. Its three cuspidal tangent planes are given by

$$u_1 = 0, \quad a'u_2^3 + bu_3^3 = 0, \qquad u_4 - \alpha u_2 - \beta u_3 = 0.$$

They intersect the tangent plane of P in its Segre tangents (thus providing a new definition for the Segre tangents) and have in common, as cusp-axis, the line l'. Thus the Green reciprocal of l coincides with the cusp axis of the quartic cone which corresponds to l in Segre's correspondence.

5. Fubini's integral invariant and Green's pseudo-normal

The integral

$$I=\int\sqrt{a'bdudv}=\int\sqrt{a'bv'}du \tag{38}$$

is invariant for all transformations of the form

$$\bar{u}=\alpha(u),\ \bar{v}=\beta(v),\ \bar{y}=\lambda(u,v)y.$$

Consequently, the value of this integral extended over an arc of a curve C on S represents a quantity intrinsically and projectively determined by this arc. If the asymptotic curves of S had not been chosen as parametric curves, the integrand of I would differ from

$$\sqrt{Ddu^2+2D'dudv+D''dv^2},$$

the square-root of the second fundamental form of S, only by a factor which makes the product a projective invariant of the arc. This factor moreover is the projective differential invariant whose vanishing is characteristic of ruled surfaces.

Fubini has proposed to introduce this integral I as a projectively defined substitute for the notion of length of arc. The extremal curves of I may then be regarded as a projectively defined substitute for the geodesics of the metric theory. These extremals are given by

$$v''+\frac{(a'b)_v}{a'b}v'^2-\frac{(a'b)_u}{a'b}v'=0, \tag{39}$$

an equation of form (4) for which

$$A=0,\ 3B=-\frac{(a'b)_v}{a'b},\ 3C=\frac{(a'b)_u}{a'b},\ D=0. \tag{40}$$

The corresponding cusp-axis is the line which joins y to the point

$$y_{uv}+\left[b'-\tfrac{1}{2}\left(\frac{a'_v}{a'}+\frac{b_v}{b}\right)\right]y_u+\left[a-\tfrac{1}{2}\left(\frac{a'_u}{a'}+\frac{b_u}{b}\right)\right]y_v; \tag{41}$$

since $B_u+C_v=0$, the corresponding cusp-axis curves and flex-ray curves form conjugate nets.

*This formula shows that the cusp-axis of a surface point with respect to the extremal curves of Fubini's integral invariant coincides with the line which Green has called the pseudo-normal, and which was defined by him in an entirely different fashion.**

* See Green's Memoir, pp. 125–127.

Green and Fubini were struck, quite independently, with the analogy which exists between this line and the normal, and both of them sketched in outline far-going theories based upon this analogy. From Green's point of view, the principal analogies are these: the pseudo-normal, like the normal, is intrinsically connected with the surface, and the curves which correspond to the developables of the pseudo-normal congruence resemble the lines of curvature, which correspond to the developables of the normal congruence, by forming also a *conjugate* net. Fubini's considerations allow us to add that both normal and pseudo-normal may be defined as cusp-axes of certain integral invariants. But if we follow Fubini in taking this latter property as a definition for the pseudo-normal, an essential gap remains to be filled. For, while Fubini has shown that the integral I is intrinsically and projectively related to an arc of a curve on S, he has not explained the nature of this relation. Green's definition, by way of contrast, although somewhat complicated, is perfectly complete. We propose to complete Fubini's definition also, by providing a geometric interpretation for the integral I.

6. Interpretation of Fubini's integral invariant

Let us select a family of hypergeodesics on the surface S, and arrange the curves of the family which pass through a given point P, of S, into pairs of conjugates, one pair of hypergeodesics for every pair of conjugate tangents. Each of these tangents contains the ray-point of P with respect to the hypergeodesic which belongs to the conjugate tangent, and we have called the line joining these ray-points the ray of P with respect to such a pair of conjugate hypergeodesics. The envelope of these rays, for all pairs of conjugate hypergeodesics P, we called the ray conic of P with respect to the selected family of hypergeodesics, and its equations we found to be

$$(33b)\quad x_4=0,\ [2x_1-(3C+2a)x_2+(3B-2b')x_3]^2+4(A-2a')(D+2b)x_2x_3=0.$$

Let C be a finite arc of a real continuous curve joining two points, A and B, of the surface S, and let (u_0, v_0) and (U, V) be the values of u and v which correspond to the points A and B. Let ϵ be a positive number, and divide the arc C by means of intermediate points $P_1, P_2, \ldots, P_{n-1}$, into n smaller arcs. Let the coördinates of P_k be (u_k, v_k), where $u_n=U$, $v_n=V$, let

$$\delta u_k=u_k-u_{k-1},\quad \delta v_k=v_k-v_{k-1},$$

and let

$$\sqrt{\delta u_k^2+\delta v_k^2}\leqq\epsilon\qquad(k=1, 2,\ldots, n-1).$$

We associate, with every point P of C, the corresponding ray conic. These conics generate a surface Σ. As n grows beyond bound and ϵ approaches the

limit zero, the line $P_{k-1}\,P_k$ approaches a tangent of C as limit. Consequently there will be two points, R_{k-1} and R'_{k-1}, among the intersections of $P_{k-1}\,P_k$ with Σ, which will tend toward the points in which the tangent to C at P_{k-1} intersects the corresponding ray conic.

Now the tangent to C at P_{k-1} intersects the corresponding ray conic (33*b*) in two points whose local coördinates are proportional to

$$(42)\quad x_1=a_{k-1}+b'_{k-1}+\tfrac{3}{2}(C_{k-1}-B_{k-1})\pm 2\sqrt{(a'_{k-1}-\tfrac{1}{2}A_{k-1})\,(b_{k-1}+\tfrac{1}{2}D_{k-1})v'_{k-1}}$$
$$x_2=1,\qquad x_3=v'_{k-1},\qquad x_4=0,$$

where we are thinking of v and $v'=dv/du$ as a function of u along the curve C, and where we have written

$$a_{k-1}=a(u_{k-1},\,v_{k-1}),\ \text{etc.}$$

The equations of $P_{k-1}P_k$ differ from those of the tangent to C at P_{k-1} only by terms of order ϵ^2. Therefore, the intersections, R_{k-1} and R'_{k-1}, of $P_{k-1}\,P_k$ with Σ will differ from the points (42), the intersections of the tangent to C at P_{k-1} with Σ, only by terms of order ϵ^2. In computing the double-ratio of the four points R_{k-1}, R'_{k-1}, P_{k-1}, P_k to terms of order ϵ, we may therefore use the values given by (42) for the coördinates of R_{k-1} and R'_{k-1}.

We find therefore

$$(43)\quad \begin{aligned}&(R_{k-1},\,R'_{k-1},\,P_{k-1},\,P_k)\\ &\quad=1+4\sqrt{(a'_{k-1}-\tfrac{1}{2}A_{k-1})\,(b_{k-1}+\tfrac{1}{2}D_{k-1})\,v'_{k-1}}\,\delta u_k+(\epsilon^2)_k,\end{aligned}$$

where $(\epsilon^2)_k$ represents a quantity of order ϵ^2.

The integrand of Fubini's integral will appear in this equation if we assume $A=D=0$, although a somewhat more general hypothesis would accomplish the same result. We shall, in fact, make the still more special assumption

$$A=D=0,\quad B_u=C_v,$$

so that the hypergeodesics which we are using become curves of a pencil of conjugate nets. Then (43) reduces to

$$(43a)\qquad (R_{k-1},\,R'_{k-1},\,P_{k-1},\,P_k)=1+4\sqrt{a'_{k-1}\,b_{k-1}\,v'_{k-1}}\,\delta u_k+(\epsilon^2)_k,$$

an equation which is fundamental for our purpose.

We have associated with every point P of the arc C a conic in the corresponding tangent plane π, namely the ray conic of P with respect to an associated pencil of conjugate nets. Let us use this conic as a basis for a definition, in Cayley's sense, of non-euclidean distances in the plane π. If M and N are two distinct points of π and if R and R' are the points of intersection of the line MN

with the ray conic of this plane, we define the non-euclidean distance of MN to be

$$\Delta_{MN} = \frac{1}{4} \log (R, R', M, N).^* \tag{44}$$

From (43) we find

$$\Delta_{P_{k-1} P_k} = \sqrt{a'_{k-1} b_{k-1} v'_{k-1}} \delta u_k + (\epsilon^2)_k.$$

Thus *we may look upon the elements of Fubini's integral as infinitesimal non-euclidean distances between neighboring points of the given arc, each of these infinitesimal distances being measured with respect to a different conic as absolute. The integral itself is the limit of a sum of such infinitesimal non-euclidean distances.* It is not, in general, itself a non-euclidean distance in the classical sense because, in the classical definition of a non-euclidean distance, the same quadric locus serves as absolute for all points of space.

Another interpretation of the Fubini integral may be obtained as follows. From (43*a*) we find

$$\begin{aligned} (R_0, R'_0, A, P_1) &= 1 + 4 \sqrt{a'_0 b_0 v'_0} \delta u_1 + (\epsilon^2)_1, \\ (R_1, R'_1, P_1, P_2) &= 1 + 4 \sqrt{a'_1 b_1 v'_1} \delta u_2 + (\epsilon^2)_2. \end{aligned} \tag{45}$$

Let π_0 denote the perspectivity which transforms

$$R_0, R'_0, P_1 \text{ into } R_1, R'_1, P_1$$

respectively, and let A' be the point of P_1P_2 which corresponds to A in this correspondence. Then we may replace the first equation of (44) by

$$(R_1, R'_1, A', P_1) = 1 + 4 \sqrt{a'_0 b_0 v'_0} \delta u_1 + (\epsilon^2)_1. \tag{46}$$

On account of the familiar double-ratio equation

$$(ABCD)\ (ABDE) = (ABCE),$$

we find from (45) and (46), by multiplication,

$$(R_1, R'_1, A', P_2) = [1 + 4 \sqrt{a'_0 b_0 v'_0} \delta u_1 + (\epsilon^2)_1]\ [1 + 4 \sqrt{a'_1 b_1 v'_1} \delta u_2 + (\epsilon^2)_2]. \tag{47}$$

The four points $R_1\ R'_1, A', P_2$ are on P_1P_2. We determine the perspectivity π_1 which projects

$$R_1, R'_1, P_2 \text{ into } R_2, R'_2, P_2$$

respectively, and denote by A'' the point of P_2P_3 which corresponds to A'.

* We may also define non-euclidean *angles* with respect to the axis cone.

We may then replace the left member of (47) by (R_2, R_2', A'', P_2). If we multiply both members of the resulting equation by those of

$$(R_2, R_2', P_2, P_3) = 1 + 4\sqrt{a_2'b_2v_2'}\delta u_3 + (\epsilon^2)_3,$$

we find

$$(48)\qquad \begin{aligned}(R_2, R_2', A'', P_3) = [1 + 4\sqrt{a_0'b_0v_0}\delta u_1 + (\epsilon^2)_1]\,[1 + 4\sqrt{a_1'b_1v_0'}\delta u_2 \\ + (\epsilon^2)_2]\,[1 + 4\sqrt{a_2'b_2v_2'}\delta u_3 + (\epsilon^2)_3].\end{aligned}$$

Clearly we may continue in this way, obtaining the equation

$$(49)\qquad (R_{n-1}, R_{n-1}', A^{(n-1)}, B) = \prod_{k=1}^{n} [1 + 4\sqrt{a_{k-1}'b_{k-1}v_{k-1}'}\delta u_k + (\epsilon^2)_k],$$

where B is the end-point of the arc under consideration, and where $A^{(n-1)}$ is a point on $P_{n-1}B$ which is derived from A, the initial point of the arc, by the sequence of perspectivities here described. It is easy to see, by familiar methods, that the product in the right member of (49) will differ from

$$\prod_{k=1}^{n} [1 + 4\sqrt{a_{k-1}'b_{k-1}v_{k-1}'}\delta u_k]$$

by terms of order ϵ at most, and that this product will approach a limit when n grows beyond bound, provided that the definite integral

$$\int_{(u_0, v_0)}^{(U, V)} \sqrt{a'bv'}du$$

exists.

Let R and R' be the points in which the tangent to C at B intersects the corresponding ray conic, and let A^* be the point on this tangent which $A^{(n-1)}$ approaches when n grows beyond bound. We find

$$(50)\qquad \kappa = (R, R', A^*, B) = \lim_{n\to\infty} \prod_{k=1}^{n} [1 + 4\sqrt{a_{k-1}'b_k\delta u_k\delta v_k}],$$

and

$$(51)\qquad \int_A^B \sqrt{a'bdudv} = \tfrac{1}{4}\log\kappa,$$

thus defining Fubini's integral as one-fourth of the logarithm of a certain cross ratio which is defined purely projectively, by a process which constitutes a multiplicative analogon of a definite integral.

In this theorem it remains to prove (51). We observe that κ is defined by (50) as a function of U and V, the coördinates of the end-point B of the arc C.

Let us assume that the integral $\int \sqrt{a'bdudv}$ still exists when the arc C is extended by adding to it the arc BB' where the coördinates of B' are $U+dU$, $V+dV$. Then

$$\kappa(U+dU, V+dV)=\kappa(U, V)\,[1+4\sqrt{a'(U,V)b(U,V)dUdV}]+(\epsilon^2),$$

and therefore

$$\frac{\log \kappa(U+dU, U+dU)-\log \kappa(U, V)}{dU}$$

will differ from

$$4\sqrt{a'(U, V)b(U, V)\frac{dV}{dU}}$$

at most by an infinitesimal of order ϵ. Consequently we have

$$\frac{d\log \kappa(U, V)}{dU}=4\sqrt{a'(U)b(V)\frac{dV}{dU}},$$

which is equivalent to (51).

The methods of this article make it possible to interpret many other integral invariants of a curve on a surface. We need merely replace the ray conic by some other conic or pair of lines.

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